Centre Number

Student Number

SCEGGS Darlinghurst

2008

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\lim_{h \to 0} \dfrac{\tan \frac{h}{2}}{h}$ **1**

(b) Find all the real values of a for which $P(x) = ax^3 - 8x^2 - 9$ is divisible by $(x - a)$. **2**

(c) Find the domain and range of $y = 5\sin^{-1}\left(\dfrac{x}{3}\right)$ **2**

(d) Solve for x: **3**

$$\frac{x}{x+2} \geq 3$$

(e) Use the substitution $u = \sqrt{x}$ to find: **3**

$$\int \frac{dx}{(1+x)\sqrt{x}}$$

(f) In New South Wales, new number plates consist of two letters followed by two digits followed by another two letters. Repetition is allowed. How many different number plates are possible? **1**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) A curve has parametric equations $x = t - 1$, $y = 1 - t^2$ **2**

Find $\dfrac{dy}{dx}$ for this curve.

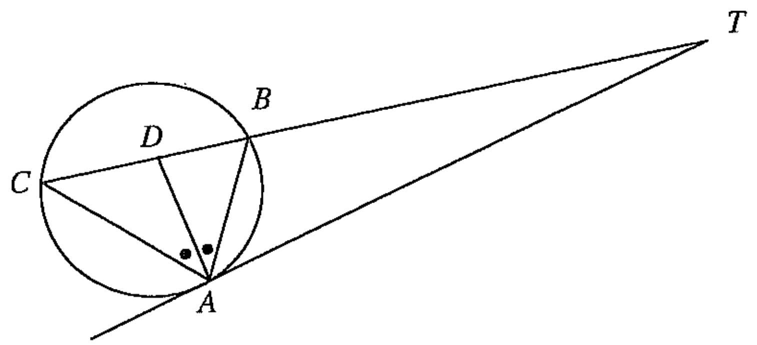
(b) (i) Write $\cos x + \sin x$ in the form $A\cos(x - \alpha)$, where $A > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$. **2**

(ii) Hence sketch the curve $y = \cos x + \sin x$ for $0 \le x \le 2\pi$. Label clearly where the curve intersects the x and y-axes. **3**

(iii) Hence, find the general solution to $\cos x + \sin x = 1$. **2**

(c) A circle passes through the points A, B and C. **3**
TA is a tangent to the circle at A.
D is a point on the secant TBC such that DA bisects $\angle BAC$.

Prove that $TA = TD$.



NOT TO SCALE

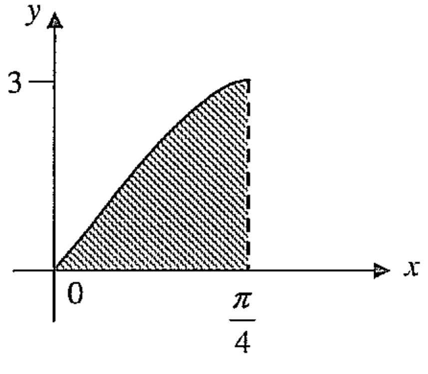
Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate $\left(5\cos^{-1} 4x\right)$ **2**

(b) Point $P(5,-2)$ divides the interval AB externally in the ratio $k:1$. **3**
If A is the point $(-1, 7)$ and B is the point $(3, 1)$, find the value of k.

(c) **3**



NOT
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SCALE

The shaded region bounded by $y = 3\sin 2x$, the x-axis and the line $x = \dfrac{\pi}{4}$ is rotated about the x-axis to form a solid of revolution.

Find the volume of the solid formed.

(d) Tonight for dinner, mum is making a lamb roast. The leg of lamb is cooked in the oven at $180°$C. Before carving the lamb, mum takes it from the oven and places it on a tray in a kitchen with a room temperature of $24°$C.
The lamb cools at a rate given by $\dfrac{dT}{dt} = k(T-24)$, where T is the temperature of the lamb in degrees Celsius after t minutes and k is a constant.

(i) Show that $T = 24 + Ae^{kt}$ satisfies the differential equation. **1**

(ii) If the temperature of the lamb falls to $160°$C after 5 minutes, find the value of A and k. **2**

(iii) Mum waits until the temperature of the lamb is $140°$C before carving it. How long does she have to wait after she takes it out of the oven? **1**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Find the size of the acute angle, to the nearest minute, between the tangents to the curve $y = e^{2x}$ at the points where $x = 0$ and $x = 1$. **3**

(b) Liam is playing with five blocks labeled with different letters *A, B, C, D, E*. He stacks three, four or five blocks on top of one another to form a vertical tower.

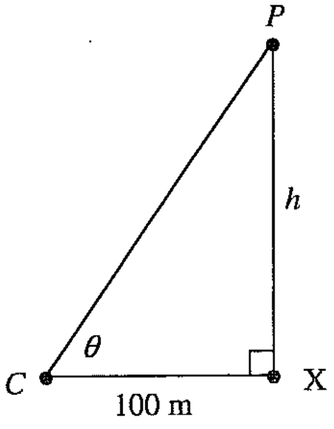
(i) How many different towers could Liam form that are three blocks high? **1**

(ii) How many different towers can Liam form in total? **2**

(iii) How many five letter towers could Liam make that contain the word *BED*, when read from top to bottom? **1**

(c) For her birthday, Betsy has decided to go skydiving. She jumps out of a plane and by the time she reaches the position, *P*, *h* metres above the ground, she is falling at a constant rate of 200 kmh^{-1}.

The point X is on horizontal ground directly below the point *P*.
Her son, Cooper, is standing at point *C*, 100 metres from X.
The angle of elevation of *P* from *C* is θ radians.



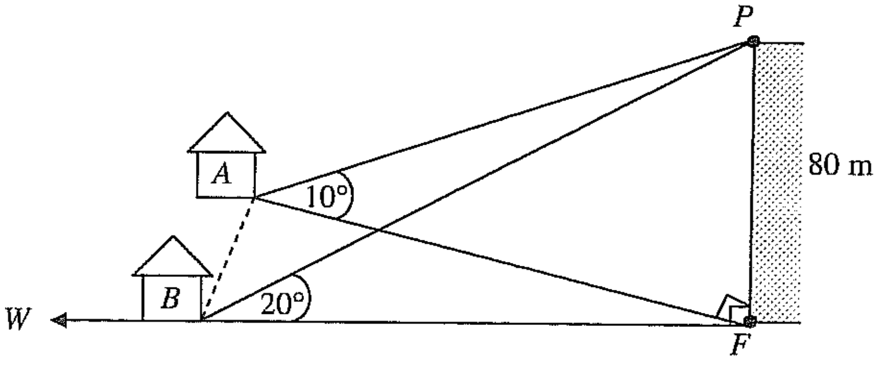
(i) Show that $\dfrac{dh}{d\theta} = \dfrac{100}{\cos^2\theta}$. **2**

(ii) Find the rate of decrease of the angle of elevation when Betsy reaches a height of 2000 metres.
(Answer in radians/second correct to 2 significant figures.) **3**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) From a lookout in the Blue Mountains on top of a vertical cliff, P, which is 80 m high, the angles of depression of two farmhouses in the Megalong Valley below are observed to be $10°$ and $20°$ respectively.

The first farmhouse, A, is northwest and the second farmhouse, B, is due west of the foot of the cliff, F.



(i) Using ΔBPF, show that $BF = 80 \tan 70°$. **1**

(ii) Show that the distance between the farmhouses **2**

$$AB = 80\sqrt{\tan^2 80° + \tan^2 70° - 2\tan 80° \tan 70° \cos 45°}$$

(iii) Hence, find AB correct to the nearest metre. **1**

(b) Use mathematical induction to prove that for all positive integers, n **3**

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Question 5 continues on page 7

Marks

Question 5 (continued)

(c) In the expansion of $(1+x)^n$, the coefficients of $x,\ x^2,\ x^3$ form an AP.

(i) Explain why **1**

$$2\binom{n}{2} = \binom{n}{1} + \binom{n}{3}$$

(ii) Hence, show that **2**

$$n^3 - 9n^2 + 14n = 0$$

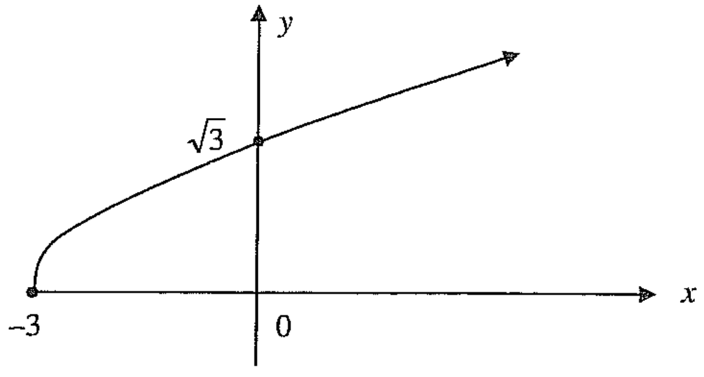
(iii) Hence, find the value of n that satisfies the above condition to form an AP. **2**

End of Question 5

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows a sketch of $y = f(x)$, where $f(x) = \sqrt{x+3}$.



(i) Copy or trace the diagram. On the same set of axes, sketch the graph of the inverse function $y = f^{-1}(x)$. **1**

(ii) State the domain of $f^{-1}(x)$. **1**

(iii) Find an expression for $y = f^{-1}(x)$ in terms of x. **1**

(iv) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at exactly one point P. Let α be the x-coordinate of P. Explain why α is a root of the equation $x - \sqrt{x+3} = 0$. **1**

(v) Take 2.5 as a first approximation for α. Use one application of Newton's method to find a second approximation for α. (Give your answer correct to 3 decimal places.) **2**

(vi) By rewriting $x - \sqrt{x+3} = 0$ as a quadratic equation, find the exact coordinates of P. **2**

(b) The polynomial $P(x) = 2x^3 + 3x^2 + kx - 2$ has roots α, β and γ.

(i) Find the value of $\alpha\beta\gamma$. **1**

(ii) If one root is the reciprocal of the other, find the third root and, hence, find the value of k. **3**

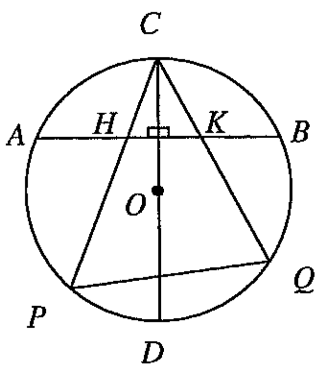
Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) After a soccer match, all eleven players must return to school. **2**
One of the players, Lucy, owns a car and takes four passengers with her.
The remaining players must return by bus.
Twins, Emma and Rachel, must return to school together.

How many different groups of five players (including Lucy) can return to school by car?

(b) **3**



NOT TO SCALE

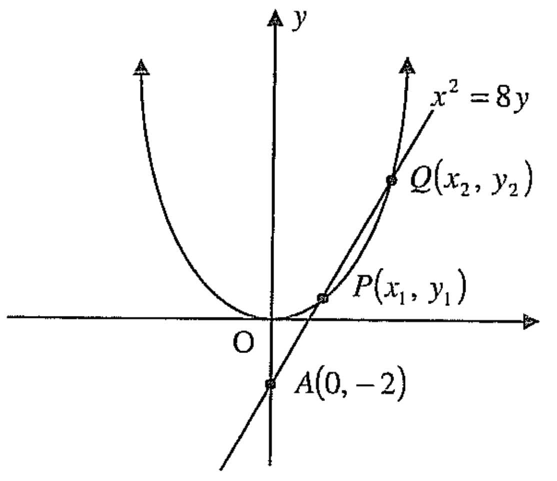
In the diagram, DC is a diameter and AB is a chord perpendicular to DC.
Chords CP and CQ cut AB at H and K respectively.

Prove that $HKQP$ is a cyclic quadrilateral.

Question 7 continues on page 10

Question 7 (continued) **Marks**

(c) The line l through the point $A(0, -2)$ with slope m meets the parabola $x^2 = 8y$ at the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.



(i) The line l has equation $y = mx - 2$. **1**
Show that x_1 and x_2 are the roots of the equation $x^2 - 8mx + 16 = 0$.

(ii) Using the fact that $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_1x_2$ **1**
and using the sum and product of the roots x_1 and x_2,
show that $(x_2 - x_1)^2 = 64(m^2 - 1)$.

(iii) Hence, show that $PQ^2 = 64\left(1 + m^2\right)\left(m^2 - 1\right)$. **2**

(iv) Find the values of m for which line l is a tangent to the parabola $x^2 = 8y$. **1**

(v) ΔSPQ is formed where S is the focus $(0, 2)$. **2**
Show that the exact area of ΔSPQ is $16\sqrt{m^2 - 1}$ units^2.

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Mathematics Ext ①. Trial HSC 2008

Calc /3

Question 1

a) $\lim_{h \to 0} \dfrac{\tan \frac{h}{2}}{h}$

$= \frac{1}{2} \lim_{h \to 0} \dfrac{\tan \frac{h}{2}}{\frac{h}{2}}$

$= \frac{1}{2} \times 1$

$= \frac{1}{2}$ ✓

Q1 in general was very poorly done. It's meant to be easy. If you struggled, you need to learn your work better because expect similar things in the HSC.

(b) divisible by $(x-a)$

$\therefore P(a) = 0$

$P(x) = ax^3 - 8x^2 - 9$

$P(a) = a^4 - 8a^2 - 9$ ✓ (=0)

$a^4 - 8a^2 - 9 = 0$

$(a^2+1)(a^2-9) = 0$

$a^2 = -1$ No real sol^n $\quad a^2 = 9$, $a = \pm 3$ ✓

You should recognise that this is a quadratic. A basic 2U question to solve this now.

(c) $y = 5\sin^{-1}\left(\frac{x}{3}\right)$

domain $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$ ✓

range $-\frac{5\pi}{2} \le y \le \frac{5\pi}{2}$ ✓

A very easy standard question. If you couldn't do this, you really need to do more practice of inverse trig. functions.

(d) $\dfrac{x}{x+2} \ge 3$ ✓

undefined for $x = -2$ ✓

Don't forget this undefined value.

$\dfrac{x}{x+2} \times (x+2)^2 \ge 3(x+2)^2$

$x(x+2) \ge 3(x+2)^2$

$0 \ge 3(x+2)^2 - x(x+2)$

factorise

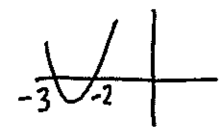
$(x+2)\left[3(x+2) - x\right] \le 0$

$(x+2)\left[3x+6-x\right] \le 0$

$(x+2)(2x+6) \le 0$

$2(x+2)(x+3) \le 0$ ✓

→ factorising here is easier than expanding



$-3 \le x \le -2$ ✓

Solution $-3 \le x < -2$

You will get a mark for correct solution of your quadratic inequality.

Calc 3

(e) $u = \sqrt{x}$

$x = u^2$

$\frac{dx}{du} = 2u$

$dx = 2u.du$

$\int \dfrac{dx}{(1+x)\sqrt{x}}$

$= \int \dfrac{2u\,du}{(1+u^2).u}$ ✓

$= \int \dfrac{2}{1+u^2}\,du$ ✓

$= 2\tan^{-1} u + C$

$= 2\tan^{-1}\sqrt{x} + C$ ✓

→ Very easy from this point on. LOOK AT THE STANDARD INTEGRALS. Why didn't you use them?

(f) $26 \times 26 \times 10 \times 10 \times 26 \times 26$

$= 26^4 \times 10^2$ ✓

There are 26 letters in the alphabet.

Question 2 Comm /3 , Reas /5

(a)

$x = t - 1$, $y = 1 - t^2$

$\frac{dx}{dt} = 1$, $\frac{dy}{dt} = -2t$ ✓

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -2t \times 1$

$= -2t$ ✓

$= -2(x+1)$

OR $y = 1 - (x+1)^2$ ✓

$\frac{dy}{dx} = -2(x+1)$ ✓

Oops! This should have been calc.

This is a standard question & reasonably well done.

There were some strange attempts to incorporate $x^2 = 4ay$!?!

(b) i) $A\cos(x - \alpha)$

$= A\cos x\cos\alpha + A\sin x\sin\alpha$

$\equiv \cos x + \sin x$

Match coefficients

$A\cos\alpha = 1$ ①

$A\sin\alpha = 1$ ②

Find α ②/①

$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$ ✓

Find A

$A^2 = 1^2 + 1^2$

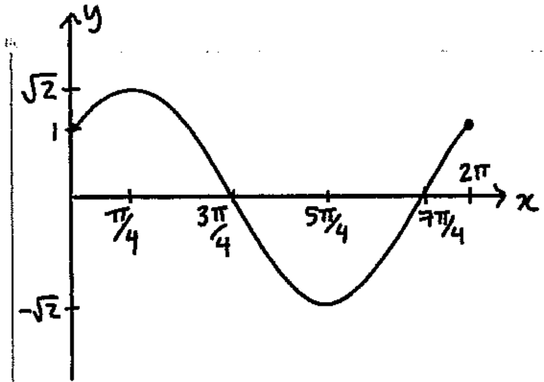
$A = \sqrt{1+1}$

$= \sqrt{2}$ ✓

$\therefore \cos x + \sin x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$

part (i) was well done but the sketch in (ii) & the general solution in (iii) not so.

ii)



✓ Correct Range/amplitude
✓ x intercepts
✓ y intercept

Comm 3

iii)

$\cos x + \sin x = 1$

$\sqrt{2}\cos(x - \pi/4) = 1$

$\cos(x - \pi/4) = 1/\sqrt{2}$

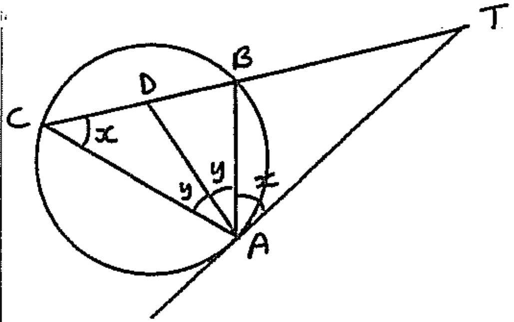
$x - \pi/4 = \pm\pi/4 + 2n\pi, \; n \in \mathbb{Z}$

$x = \pi/2 + 2n\pi, \; 2n\pi$ ✓ ✓

Reas 2

This is a simple general solution that could even be read, or at the least checked, from the sketch in (ii)

(c)



$\angle TAB = \angle ACB = x$
(angle between tangent and chord is equal to angle in alternate segment.) ✓

DA bisects $\angle BAC$
$\angle CAD = \angle BAD = y$

$\angle TDA = \angle DCA + \angle DAC$
$= x + y$
(Exterior $\angle$ of a Δ = sum of 2 opposite interior $\angle$s.) ✓

$\angle DAT = \angle DAB + \angle BAT$
$= x + y$

$\therefore \angle TDA = \angle DAT$

$\therefore TD = TA$
(equal sides opposite equal angles in an isosceles Δ are equal.) ✓

Reas 3

It is always a good idea to reproduce the diagram on your answer page.

Also too many people are introducing new letters here there & everywhere and you can't expect the marker to be able to read your mind.
If you are going to call something 'E' have it on your diagram or explain where it is!

And be very careful of typos! They can & will lose you marks.

Question 3.

Calc /6
Reas /3

(a)

$y = 5\cos^{-1} 4x$

$y' = 5 \times \frac{-1}{\sqrt{1-(4x)^2}} \times 4$ ✓

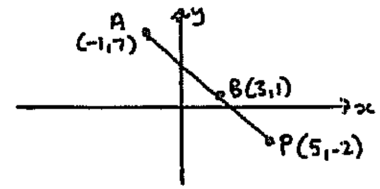
$= \frac{-20}{\sqrt{1-16x^2}}$ ✓

$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2-x^2}}$

$y' = -\frac{5}{\sqrt{\frac{1}{16}-x^2}}$

Calc 3 Learn the rules!!

(b)



$AB = \sqrt{(3+1)^2 + (1-7)^2}$
$= \sqrt{4^2 + (-6)^2}$
$= \sqrt{16+36}$
$= \sqrt{52}$
$= 2\sqrt{13}$

$PB = \sqrt{(5-3)^2 + (-2-1)^2}$
$= \sqrt{2^2 + -3^2}$
$= \sqrt{4+9}$
$= \sqrt{13}$ ✓

$AP : PB$
$3\sqrt{13} : \sqrt{13}$
$3 : 1$ ✓
P divides AB externally in the ratio 3:1

$\therefore k = 3$ ✓

You don't have to use any formulas if you do it this way.

Using the formula works too.

Reas 3

11/08/2008

c) $y = 3\sin 2x$

About x axis.

$V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^{\pi/4} (3\sin 2x)^2\,dx$

$= \pi\int_0^{\pi/4} 9\sin^2 2x\,dx$

$= 9\pi\int_0^{\pi/4} \frac{1}{2}(1-\cos 4x)\,dx$ ✓

$= \frac{9\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/4}$ ✓

$\frac{9\pi}{2}\left\{\frac{\pi}{4} - \frac{1}{4}\sin\pi - 0 - \frac{1}{4}\sin 0\right\}$

$= \frac{9\pi}{2} \times \frac{\pi}{4}$

$= \frac{9\pi^2}{8}\ u^2$ ✓

If you got this wrong you really don't know your work well enough.

Highlight key words. You must notice that it asks for Volume not area.

Learn the rules and expect to use them.

$\int \sin^2 x\,dx =$

$\int \cos^2 x\,dx =$

Calc 3

d) i) $T = 24 + Ae^{kt}$

$\frac{dT}{dt} = k.Ae^{kt}$

$= k(T-24)$ ✓

The equation is satisfied ☺

A standard question. Learn how to do it this way.

An easy mark.

Calc 1

11/08/2008

ii) $t=0 \quad T=180°C$

$t=5 \quad T=160°C$

$T = 24 + Ae^{kt}$

$180 = 24 + Ae^0$

$\therefore A = 156$ ✓

$T = 24 + 156e^{kt}$

when $t=5$ $T=160$

$160 = 24 + 156e^{5k}$

$136 = 156e^{5k}$

$e^{5k} = \frac{136}{156}$

$5k = \ln\left(\frac{136}{156}\right)$

$k = \frac{1}{5}\ln\left(\frac{136}{156}\right)$

$k \doteqdot -0.02744$ ✓

↳ Store this in your calculator.

iii) $T = 24 + 156e^{kt}$

when $T = 140°C$

$140 = 24 + 156e^{kt}$

$116 = 156e^{kt}$

$e^{kt} = \frac{116}{156}$

$kt = \ln\left(\frac{116}{156}\right)$

$t = \frac{1}{k}\ln\left(\frac{116}{156}\right)$

↑ use the stored value from your calculator

$t \doteqdot 10.79677...$

$t \doteqdot 10$ min 48 sec ✓

Very well done.

Question 4 Calc /6, Reas /4

a) $y = e^{2x}$

$y' = 2e^{2x}$

when $x=0$ $m_1 = 2e^0$
$= 2$

when $x=1$ $m_2 = 2e^2$ ✓

Acute angle

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{2e^2 - 2}{1 + 2 \times 2e^2}\right|$ ✓

$= \left|\dfrac{2e^2 - 2}{1 + 4e^2}\right|$

$\theta = 22^\circ 42'$ ✓

(Calc 1)

You need to know this formula! I can't believe how many people got it wrong.

b) i) $5 \times 4 \times 3 = 60$ ways ✓

ii) 3 blocks $= 5 \times 4 \times 3 = 60$
4 blocks $= 5 \times 4 \times 3 \times 2 = 120$ ✓
5 blocks $= 5! = \underline{120}$ ✓
300

iii) [BED] [A] [C]
↑
Treat as one group

No ways $= 3! = 6$ ways ✓

(Reas 4)

The biggest mistake was people not reading the question.

c) i) $\tan\theta = \dfrac{h}{100}$

$h = 100 \tan\theta$ ✓

$\dfrac{dh}{d\theta} = 100 \sec^2\theta$

$\dfrac{dh}{d\theta} = \dfrac{100}{\cos^2\theta}$ ✓

ii) Find $\dfrac{d\theta}{dt}$

$\dfrac{d\theta}{dt} = \dfrac{d\theta}{dh} \times \dfrac{dh}{dt}$

$= \dfrac{\cos^2\theta}{100} \times \dfrac{2000}{36}$

$= \dfrac{\cos^2 1.52}{100} \times \dfrac{2000}{36}$ ✓

$\doteqdot 0.0014$ rad/sec

(2 sig. figs.)

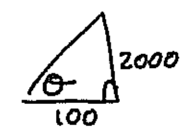
(Calc 5)

Part (i) should have been so easy but people were still making a mess of it

$\dfrac{dh}{dt} = 200$ km/h

$= \dfrac{200000}{3600}$ m/s ✓

when $h = 2000$



$\tan\theta = 20$ ✓

$\theta \doteqdot 1.52$ rad

People just made a mess of this part!
Particularly:
- finding θ
- not considering units!
- not being able to calculate an answer!

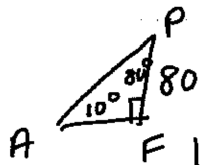
Reas /4

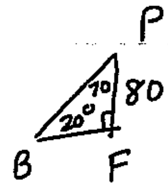
Comm /3

11/08/2008

Question 5

(a) i)





$\tan 80° = \frac{AF}{80}$

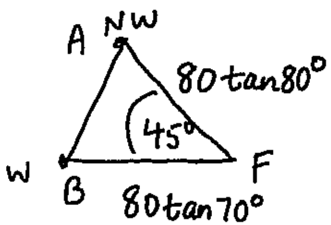
$AF = 80 \tan 80°$

$\tan 70° = \frac{BF}{80}$

$BF = 80 \tan 70°$ ✓

This part was very well done in this question.

ii) Birds Eye view



✓

$AB^2 = AF^2 + BF^2 - 2AF.BF.\cos F$

$= 80^2\tan^2 80° + 80^2\tan^2 70°$

$- 2 \times 80\tan 80 . 80\tan 70 \cos 45$ ✓

$AB^2 = 80^2\left(\tan^2 80° + \tan^2 70° - 2\tan 80° \tan 70° \cos 45°\right)$

$AB = 80\sqrt{\tan^2 80° + \tan^2 70° - 2\tan 80 \tan 70 \cos 45°}$

$AB \doteq 336$ m ✓

Check your calculator work.

11/08/2008

(b) Prove that $\sum_{r=1}^{n}(2r-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Prove true for n=1

$LHS = (2-1)^2 = 1$

$RHS = \frac{1 \times (2-1) \times (2+1)}{3} = \frac{1 \times 1 \times 3}{3} = 1$ ✓

LHS=RHS ∴ True for n=1

This part was easy! Everyone got it correct.

Assume true for n=k

$\sum_{r=1}^{k}(2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

↑ must have sigma notation or be written out as sum of a series.

(*) You must include the Σ or write out the whole series.

Prove true for n=k+1

$\sum_{r=1}^{k+1}(2(k+1)-1)^2 = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

$= \frac{(k+1)(2k+1)(2k+3)}{3}$

$LHS = \underbrace{1^2+3^2+5^2+\dots+(2k-1)^2}+(2k+1)^2$

$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$ By the assumption ✓

$= \frac{(2k+1)}{3}\left[k(2k-1) + 3(2k+1)\right]$

$= \frac{(2k+1)}{3}\left[2k^2 - k + 6k + 3\right]$

$= \frac{(2k+1)}{3}\left[2k^2 + 5k + 3\right]$

$= \frac{(2k+1)}{3}(2k+3)(k+1)$ ✓

$= RHS$

Reas 2

Conclusion If the statement is true for n=k, then it is true for n=k+1. Since it is true for n=1, then it is true for n=2 and so on. By the principle of mathematical induction, it is true for all positive integers, n.

(5) c) $(1+x)^n$

Coefficients of x, x^2, x^3 are $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}$

If they Form an AP

$T_2 - T_1 = T_3 - T_2$ ✓

$\binom{n}{2} - \binom{n}{1} = \binom{n}{3} - \binom{n}{2}$

$2\binom{n}{2} = \binom{n}{1} + \binom{n}{3}$

Not as hard as you may have thought.

Be prepared in Ext(1) to know all the 2u course.

(Comm 1)

---

ii) $\frac{2n!}{2!(n-2)!} = \frac{n!}{1!(n-1)!} + \frac{n!}{3!(n-3)!}$

$n(n-1) = n + \frac{n(n-1)(n-2)}{3 \times 2}$ ✓

$6(n^2 - n) = 6n + n(n^2 - 3n + 2)$

$6n^2 - 6n = 6n + n^3 - 3n^2 + 2n$ ✓

$n^3 - 9n^2 + 14n = 0$

Unrolling the factorials is easier than finding a common denominator.

If you divide n! you will lose a solution.

(Reas 2)

$n(n^2 - 9n + 14) = 0$

$n(n-7)(n-2) = 0$

$n = 0$ (not valid) $\quad n = 7 \quad n = 2$ ✓ (not valid because $(1+x)^2 = 1 + 2x + x^2$ not enough terms.)

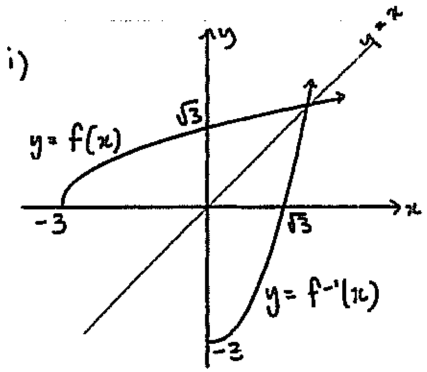
$\therefore n = 7$ ✓

A very easy first mark. Solve the equation that's printed on the page.

(Comm 2)

Question 6 Comm 2

a) i)



(Comm 1)

If you have two graphs drawn on the same set of axes make sure you label them!

---

ii) Domain of $f^{-1}(x)$

$x \geq 0$ ✓

---

iii) $y = \sqrt{x-3}$ original function

Interchange x & y

$x = \sqrt{y-3}$ inverse

$x^2 = y - 3$

$y = x^2 + 3$ ✓

$f^{-1}(x): \; y = x^2 - 3$ where $x \geq 0$

You must make y the subject if they ask for $y = f^{-1}(x)$

---

iv) $y = f(x)$ intersects the line $y = x$ once in the first quadrant, thus $y = f(x)$ intersects $y = f^{-1}(x)$ at this same (single) point in the first quadrant — the point P

$\therefore$ for pt of int, P, between $y = f(x)$ & $y = f^{-1}(x)$ solve $y = f(x)$ & $y = x$ simultaneously

$\sqrt{x-3} = x$

$x - \sqrt{x-3} = 0$

$\therefore \alpha$, the x coord of P is a solution to this equation.

(Comm 1)

It is NOT correct to say $y = f(x)$ & $y = f^{-1}(x)$ always intersect on the line $y = x$. Be very careful with your wording

(Counter example: $f(x) = -x^3$)

v)

$$P(x) = x - \sqrt{x+3}$$
$$= x - (x+3)^{1/2}$$

$$P'(x) = 1 - \tfrac{1}{2}(x+3)^{-1/2} \cdot 1$$
$$= 1 - \frac{1}{2\sqrt{x+3}}$$

$$P(2.5) = 2.5 - \sqrt{2.5+3}$$
$$= 2.5 - \sqrt{5.5}$$
$$= 0.1547\ldots$$

$$P'(2.5) = 1 - \frac{1}{2\sqrt{2.5+3}}$$
$$= 1 - \frac{1}{2\sqrt{5.5}}$$
$$= 0.7867993\ldots$$

By Newton's method, a better approximation would be

$$x_2 = x_1 - \frac{P(x_1)}{P'(x_1)}$$ ✓
$$= 2.5 - \frac{0.1547\ldots}{0.7867\ldots}$$
$$\doteqdot 2.303$$ ✓

You need to approximate the solution to $x - \sqrt{x+3} = 0$, so your function will be $P(x) = x - \sqrt{x+3}$ <u>NOT</u> $f(x) = \sqrt{x+3}$

(b) $P(x) = 2x^3 + 3x^2 + kx - 2$

i) $\alpha\beta\gamma = -d/a$
$= - -2/2$
$= 1$ ✓

These were easy marks to pick up

ii) Let the roots be $\alpha, \frac{1}{\alpha}, \beta$

Product of the roots
$\Rightarrow \alpha \cdot \frac{1}{\alpha} \cdot \beta = 1$

$\Rightarrow \beta = 1$ ✓

$\therefore P(1) = 0$ ✓
$\Rightarrow 2 + 3 + k - 2 = 0$
$k = -3$ ✓

This was definitely the easiest way once you found the third root.

OR/ Sum of roots $= -b/a$
$\Rightarrow \alpha + \frac{1}{\alpha} + \beta = -3/2$
$\alpha + \frac{1}{\alpha} = -\frac{5}{2}$ ✓

Sum of roots two @ a time $= c/a$
$\Rightarrow \alpha \cdot \frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha} \cdot \beta = \frac{k}{2}$
$\Rightarrow 1 + \beta\left(\alpha + \frac{1}{\alpha}\right) = \frac{k}{2}$
$\Rightarrow 1 + 1\left(-\frac{5}{2}\right) = \frac{k}{2}$
$\Rightarrow k = -3$ ✓

Question 7

Reas/7 Comm/3

(a) Twins return by car

1 × 1 × 1 × $\binom{8}{2}$
Lucy Emma Rachel choose 2 other students

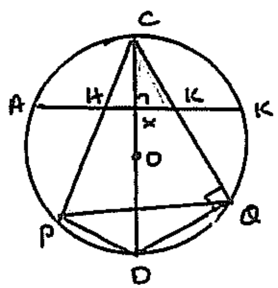
Twins don't go by car → 2 go by bus
→ 8 left to pick from

1 × $\binom{8}{4}$
Lucy Choose another 4 to go in the car.

Total number of ways = $\binom{8}{2} + \binom{8}{4}$

This question was quite well done. Good work ü

(b)



Contruction: Construct PD and DQ

In ΔCQD
$\angle CQD = 90°$
($\angle$ in a semicircle is 90°)

In ΔCQD and ΔCXK
$\angle CQD = \angle CXK = 90°$
$\angle C$ is common
∴ ΔCQD ||| ΔCXK (Equiangular)

There's more than one way to do this question but the reasons and steps you write must include some circle geometry facts and be leading towards the required conclusion.

∴ $\angle CDQ = \angle CKX$ (matching angles in similar triangles)

Let $\alpha = \angle CDQ$ and $\angle CKX$

Now $\angle CPQ = \angle CDQ = \alpha$ (angles in the same segment are equal)

∴ $\angle CKX = \angle CPQ = \alpha$
∴ HKQP is a cyclic quadrilateral
(The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle).

(comm3)

i) The points $P(x_1,y_1)$ and $Q(x_2,y_2)$ are the points of intersection of

$y = mx - 2$ ①
$x^2 = 8y$ ②

The x values, $x_1 x_2$ satisfy the equation
Sub. ① into ②

$$x^2 = 8(mx-2)$$
$$x^2 = 8mx - 16$$
$$x^2 - 8mx + 16 = 0$$

✓

ii) Sum $x_1 + x_2 = -b/a$
$= 8m/1$

Product $x_1 x_2 = c/a$
$= 16/1$

$$(x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2$$
$$= (8m)^2 - 4\times16$$
$$= 64m^2 - 64$$
$$= 64(m^2-1)$$

✓

Note $(x_1-x_2)^2$ is exactly the same result as $(x_2-x_1)^2$

iii) $PQ^2 = (x_2-x_1)^2 + (y_2-y_1)^2$
from part ii)
$= 64(m^2-1) + (y_2-y_1)^2$

Since the points PQ are on line l
Now $y_2 = mx_2 - 2$ $y_1 = mx_1 - 2$ ✓

$$= 64(m^2-1) + (mx_2 - 2 - (mx_1 - 2))^2$$
$$= 64(m^2-1) + (mx_2 - mx_1)^2$$ ✓
$$= 64(m^2-1) + m^2(x_2-x_1)^2$$
$$= 64(m^2-1) + m^2\, 64(m^2-1)$$
$$= 64(m^2-1)(1+m^2)$$

→ You could also do this on the parabola. $y = \frac{x^2}{8}$
↓
$(y_2-y_1)^2 = \left(\frac{x_2^2 - x_1^2}{8}\right)^2$
etc.
be careful of algebra.

(Reas 2)

iv) Tangent if $PQ^2 = 0$

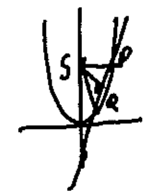
$$64(m^2-1)(m^2+1) = 0$$

$m^2 - 1 = 0$ $m^2 + 1 = 0$
$m = \pm 1$ No solution. ✓

Or use the discriminant
$\Delta = 0$
$b^2 - 4ac = 0$
$64m^2 - 64 = 0$
$64m^2 = 64$
$m^2 = 1$
$m = \pm 1$

(Reas 1)

v)



$$PQ = \sqrt{64(m^2-1)(m^2+1)}$$
$$= 8\sqrt{(m^2-1)(m^2+1)}$$

line l $y = mx - 2$
$mx - y - 2 = 0$
Perpendicular distance from l to $S(0,2)$

$$d = \frac{|0 - 2 - 2|}{\sqrt{m^2+1}}$$
$$= \frac{|-4|}{\sqrt{m^2+1}}$$
$$= \frac{4}{\sqrt{m^2+1}}$$ ✓

Area $\Delta SPQ = \frac{1}{2} \times 8\sqrt{(m^2-1)(m^2+1)} \times \frac{4}{\sqrt{m^2+1}}$

$$= 16\sqrt{m^2-1}$$ ✓

You can do this part without having done the previous parts. Just use a bit of logic.

(Reas 2)